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Quantum Clebsch–Gordan coefficients for non-generic q values

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Abstract. In the decomposition of a coproduct in the direct product of two irreducible representation spaces of the quantum enveloping algebra $U_q sl(2)$, the mixed representation, reducible but indecomposable, appears when q is a root of unity. In this paper, the necessary and sufficient condition for mixture of two irreducible representations are presented, the quantum Clebsch–Gordan coefficients which are neither all vanishing nor divergent for the non-generic q values are defined, and the method for computing the new states are discussed in some detail.

1. Introduction

The quantum enveloping algebras $U_q\mathscr{G}$ were firstly presented [1] as a tool for solving the Yang–Baxter equation [2] which plays a crucial role in some completely integrable statistical systems, and now they draw the increasing interests of theoretical physicists and mathematicians. The properties of $U_q\mathscr{G}$ for the generic q values are studied quite well, but the theory for the non-generic values where q is a root of unity is in the preliminary stage [3–5]. However, all the irreducible representations (IR) of $U_q\, sl(2)$ for the non-generic q values are known very well.

Recently, from the study of an XXZ spin chain model [6, 7], the structure of the type I representations which are reducible but indecomposable was studied. The necessary condition for appearance of the type I representations was given, but there are several problems that should be studied further. Among them, the sufficient condition, the new states, and the quantum Clebsch–Gordan (qCG) coefficients for the non-generic q values are the most urgent ones. In this paper, we are going to study those problems in the decomposition of a coproduct in the direct product of two irreducible representation (IR) spaces for $U_q\, sl(2)$ in detail.

A state of one IR in the decomposition of a coproduct may coincide with a state of another IR when q goes to a non-generic value. When it occurs, we call that two states are degenerate and two IRs are mixed. In this paper we will present the mixed condition of two IRs in the decomposition of a coproduct and the method for computing the new states appearing due to the degenerate states. The qCG coefficients for the non-generic q values will be studied in some detail.

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The plan of this paper is as follows. In section 2 we will show some formulas for the non-generic q values which are useful for the later computations. In section 3, we will present a theorem for the mixed condition under which the relevant quantum Clebsch–Gordan coefficients do coincide with each other for the non-generic q value. In this case, some factors in the numerator or denominator of the qCQ coefficients may be vanishing, so a new definition is needed to rule out the vanishing or divergent factors, and also given in section 3. The proof of the theorem are given in section 4. Since some states are degenerate, the method for computing the new states, which span together with the old states a type I representation, will be discussed in section 5.

2. Formulas for the non-generic q values

For a given integer p, q_0, called a non-generic value, is defined as

$$q_0^p=\lambda=\begin{cases}1 \text{ or } -1 & \text{when } p \text{ is odd}\\ -1 & \text{when } p \text{ is even.}\end{cases} \tag{1}$$

Let

$$[m]=\frac{q^m-q^{-m}}{q-q^{-1}}. \tag{2}$$

When $q=q_0$ we denote $[m]$ as $[m]_0$. Obviously,

$$[np]_0=0 \qquad [\alpha]_0\neq 0 \qquad 0<\alpha<p. \tag{3}$$

In this paper, if without a special notification, a small latin letter, for example n or p etc denotes a non-negative integer, and a small greek letter, for example α, denotes a non-negative integer less than p: $0\leqslant\alpha<p$.

It is easy to check the following useful formulas for $q=q_0$

$$[np+\alpha]_0=\lambda^n[\alpha]_0 \qquad [np-\alpha]_0=-\lambda^n[\alpha]_0. \tag{4}$$

From (4) we can show

$$\begin{bmatrix}p-1\\ \alpha\end{bmatrix}_0=(-\lambda)^\alpha \tag{5}$$

where

$$\begin{bmatrix}n\\ m\end{bmatrix}=\frac{[n][n-1]\ldots[n-m+1]}{[m]!}=\frac{[n]!}{[m]!\,[n-m]!}$$
$$[m]!=[m][m-1]\ldots[1] \qquad [0]!=1 \qquad [-n]!\to\infty \tag{6}$$

and the subscript 0 denotes $q=q_0$.

In terms of the factorization method

$$\frac{[np]}{[p]}=\frac{q^{np}-q^{-np}}{q^p-q^{-p}}=q^{(n-1)p}+q^{(n-3)p}+\ldots+q^{-(n-1)p} \tag{7}$$

we have

$$\lim_{q\to q_0}\frac{[np]}{[p]}=n\lambda^{n-1} \qquad \text{and} \qquad \lim_{q\to q_0}\frac{[np]}{[mp]}=\frac{n}{m}\lambda^{n-m}. \tag{8}$$

Generally, we obtain

$$\lim_{q\to q_0}\begin{bmatrix}np+\alpha\\ mp+\beta\end{bmatrix}=\lambda^{\alpha m+\beta n+pm(n-1)}\begin{pmatrix}n\\ m\end{pmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_0 \qquad \text{when } \alpha\geqslant\beta \tag{9a}$$

$$\lim_{q\to q_0} \frac{\begin{bmatrix} np+\alpha \\ mp+\beta \end{bmatrix}}{[p]}$$

$$= (-1)^{\beta-\alpha-1}\lambda^{\alpha m+\beta n+pm(n-1)-1}\binom{n}{m}(n-m)$$

$$\times\left\{\begin{bmatrix} \beta-1 \\ \alpha \end{bmatrix}_0 [\beta]_0\right\}^{-1} \qquad \text{when } \alpha<\beta. \tag{9b}$$

Now, we are going to show formulas on derivatives with respect to q denoted by prime. Because

$$[np+\alpha]' = \left(\frac{q^{np+\alpha}-q^{-np-\alpha}}{q-q^{-1}}\right)'$$
$$= (q^2-1)^{-1}\{(np+\alpha)(q^{np+\alpha}+q^{-np-\alpha})-[np+\alpha][2]\}$$

and

$$\lim_{q\to q_0}\frac{q^{p-\alpha}+q^{-p+\alpha}}{q^{p-\alpha}-q^{-p+\alpha}} = -\lim_{q\to q_0}\frac{q^{\alpha}+q^{-\alpha}}{q^{\alpha}-q^{-\alpha}} \qquad 1\leqslant\alpha\leqslant p-1 \tag{10}$$

we have

$$\lim_{q\to q_0}\frac{[np\pm\alpha]'}{[np\pm\alpha]} = \lim_{q\to q_0}\frac{\mathrm{d}}{\mathrm{d}q}\ln[np\pm\alpha] = (q_0^2-1)^{-1}\{(\pm np+\alpha)[2\alpha]_0/[\alpha]_0^2-[2]_0\} \tag{11a}$$

$$\lim_{q\to q_0}[np]' = \frac{2np\lambda^n}{q_0^2-1}. \tag{11b}$$

From (11) we have

$$\lim_{q\to q_0}\sum_{\nu=1}^{\beta}\left(\frac{[np\pm\nu]'}{[np\pm\nu]}-\frac{[mp+\nu]'}{[mp+\nu]}\right) = (\pm n-m)p(q_0^2-1)^{-1}\sum_{\nu=1}^{\beta}[2\nu]_0/[\nu]_0^2 \tag{12}$$

(12) becomes vanishing when $\beta=p-1$ owing to (10). Furthermore, because of (7) we have

$$\lim_{q\to q_0}\left(\frac{[np]}{[p]}\right)' = 0$$
$$\lim_{q\to q_0}\left(\frac{[np]}{[mp]}\right)' = \lim_{q\to q_0}\left(\frac{q^{(n-1)p}+q^{(n-3)p}+\ldots+q^{-(n-1)p}}{q^{(m-1)p}+q^{(m-3)p}+\ldots+q^{-(m-1)p}}\right)' = 0. \tag{13}$$

From (12) and (13), we obtain the following useful formulas:

$$\lim_{q\to q_0}\frac{\mathrm{d}}{\mathrm{d}q}\ln\begin{bmatrix} np-1 \\ mp+\beta \end{bmatrix} = \lim_{q\to q_0}\sum_{\nu=1}^{\beta}\left(\frac{[(n-m)p-\nu]'}{[(n-m)p-\nu]}-\frac{[mp+\nu]'}{[mp+\nu]}\right)$$
$$= -np(q_0^2-1)^{-1}\sum_{\nu=1}^{\beta}[2\nu]_0/[\nu]_0^2 \tag{14}$$

$$\lim_{q\to q_0}\frac{\mathrm{d}}{\mathrm{d}q}\ln\begin{bmatrix} np+\alpha \\ mp+\beta \end{bmatrix}$$
$$= \lim_{q\to q_0}\left\{\sum_{\nu=1}^{\alpha}\frac{[np+\nu]'}{[np+\nu]}+\sum_{\nu=1}^{p-\alpha+\beta-1}\frac{[(n-m+1)p-\nu]'}{[(n-m+1)p-\nu]}-\sum_{\nu=1}^{\beta}\frac{[mp+\nu]'}{[mp+\nu]}-\sum_{\nu=1}^{p-1}\frac{[\nu]'}{[\nu]}\right\}$$
$$= (q_0^2-1)^{-1}\left\{\sum_{\nu=\alpha-\beta+1}^{\alpha}\frac{(np-mp+\nu)[2\nu]_0}{[\nu]_0^2}+mp\sum_{\nu=\beta+1}^{\alpha}\frac{[2\nu]_0}{[\nu]_0^2}-\sum_{\nu=1}^{\beta}\frac{\nu[2\nu]_0}{[\nu]_0^2}\right\} \tag{15a}$$

where $\alpha \geqslant \beta$, and

$$\lim_{q\to q_0} \frac{d}{dq} \ln\left\{\begin{bmatrix} np+\alpha \\ mp+\beta \end{bmatrix} \Big/ [p]\right\}$$

$$= \lim_{q\to q_0}\left\{\sum_{\nu=1}^{\alpha} \frac{[np+\nu]'}{[np+\nu]} + \sum_{\nu=1}^{\beta-\alpha-1} \frac{[(n-m)p-\nu]'}{[(n-m)p-\nu]} - \sum_{\nu=1}^{\beta} \frac{[mp+\nu]'}{[mp+\nu]}\right\}$$

$$= (q_0^2-1)^{-1}\left\{\sum_{\nu=\beta-\alpha}^{\beta} \frac{(np-mp-\nu)[2\nu]_0}{[\nu]_0^2} - np\sum_{\nu=\alpha+1}^{\beta} \frac{[2\nu]_0}{[\nu]_0^2} + \sum_{\nu=1}^{\alpha} \frac{\nu[2\nu]_0}{[\nu]_0^2} + [2]_0\right\} \tag{15b}$$

where $\alpha < \beta$.

Hereafter we will say 'when q goes to q_0' or 'when $q = q_0$' to replace the limit symbol $\lim_{q\to q_0}$.

3. Mixed condition

For the generic q values, the coproduct $\Delta_q^{j_1 j_2}$ in the direct product of two IR spaces of $U_q\,sl(2)$ is a reducible representation and can be reduced by the quantum Clebsch–Gordan matrix which was computed explicitly in [8, 9].

$$\sum_{m'} (\Delta_q^{j_1 j_2})_{m(M-m)m'(M'-m')} (C_q^{j_1 j_2})_{m'(M'-m')JM'} = (C_q^{j_1 j_2})_{m(M-m)JM} (D_q^J)_{MM'} \tag{16}$$

where $J = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$. The states in the IR spaces can be combined by qCG coefficients

$$|JM\rangle = \sum_m |j_1 m\rangle |j_2 (M-m)\rangle (C_q^{j_1 j_2})_{m(M-m)JM}. \tag{17}$$

When q goes to the non-generic value q_0 (see (1)), some states $|J'M\rangle$ and $|JM\rangle$ may coincide with each other. In this section we are going to show the mixed condition.

For the non-generic q values, the normalization of a state is not important because some states may be nilpotent (a zero norm). In this case Lusztig's representation [3] may be more convenient:

$$\begin{aligned} &e|j(m-1)\rangle = [j+m]|jm\rangle \qquad e|jj\rangle = 0 \\ &f|j(m+1)\rangle = [j-m]|jm\rangle \qquad f|j-j\rangle = 0 \\ &h|jm\rangle = 2m|jm\rangle \\ &\langle jm|jm\rangle = \begin{bmatrix} 2j \\ j+m \end{bmatrix}. \end{aligned} \tag{18}$$

In this representation, the qCG coefficients are as follows [8, 9]

$$\begin{aligned} (C_q^{j_1 j_2})_{m(M-m)JM} &= A_J^{-1} q^{-(j_1+j_2-J)(j_1+j_2+J+1)/2 + mj_2 - (M-m)j_1} \Delta(j_1 j_2 J) \left\{\frac{[2J+1]!}{[2j_1]![2j_2]!}\right\}^{1/2} \\ &\times [j_1+m]![j_1-m]![j_2+M-m]![j_2-M+m]! \sum_n (-1)^n q^{n(J+j_1+j_2+1)} \\ &\times \{[n]![j_1-m-n]![j_2+M-m-n]![j_1+j_2-J-n]! \\ &\times [J-j_1-M+m+n]![J-j_2+m+n]!\}^{-1} \end{aligned} \tag{19}$$

$$\Delta(j_1 j_2 J) = \left\{\frac{[j_1+j_2-J]![j_1-j_2+J]![-j_1+j_2+J]!}{[j_1+j_2+J+1]!}\right\}^{1/2}$$

where A_J is introduced to avoid divergence or all vanish of qCG coefficients such that the state $|JM\rangle$ exists. The explicit form of A_J will be given later.

The mixed condition of two IRs $D_q^{J'}$ and D_q^J is a condition under which, when $q=q_0$, $|J'M\rangle = c|JM\rangle$, where c is a constant. Without loss of generality, we assume that $j_1 \geqslant j_2$ and $J'>J$. Since $e|JJ\rangle=0$ and $\langle J'J|JJ\rangle=0$ for the generic q values, we obtain from $|J'J\rangle = c|JJ\rangle$ when q goes to q_0,

$$\lim_{q\to q_0}\langle J'J|J'J\rangle = \lim_{q\to q_0} A_{J'}^{-2}\begin{bmatrix} 2J' \\ J'+J\end{bmatrix}=0$$

$$\lim_{q\to q_0} e|J'J\rangle = \lim_{q\to q_0}[J'+J+1]|J'(J+1)\rangle = 0$$

that just is the necessary condition for mixture obtained in the XXZ spin chain model [6, 7]:

$$J'+J+1=0 \qquad \mathrm{mod}\ p. \tag{20}$$

Introduce the following notations:

$$0 \leqslant j_1-j_2 = sp+\gamma \qquad 2j_2 = up+\omega \tag{21}$$

$$J'=tp+\eta \qquad J=(l-t)p-\eta-1<J' \tag{22a}$$

where both γ and η are integers or half of odd integers spontaneously because j_1 and j_2 may be an integer or half of an odd interger, respectively. Now, the mixed condition is shown in the theorem.

Theorem. In the decomposition of the coproduct $\Delta_q^{j_1j_2}$, the necessary and sufficient condition for mixture of two IRs $D_q^{J'}$ and D_q^J is as follows:

$$l=\begin{cases}2t & \text{if } \eta<\gamma<p-\eta\\ 2t+1 & \text{if } (p-1)/2<\eta<\gamma \text{ or } \gamma<(p-1)/2<\eta<p-\gamma.\end{cases} \tag{22b}$$

We will prove the theorem in the next section. Here, we are going to give some remarks.

At first, it is easy to check that (22) are equivalent to the following condition

$$J'=tp+\eta \qquad J=(l-t)p-\eta-1 \qquad 0\leqslant \eta<(p-1)/2$$

$$l=\begin{cases}2t & \text{if } \eta<\gamma<p-\eta\\ 2t+1 & \text{if } \eta\geqslant\gamma \text{ or } p-\eta\leqslant\gamma.\end{cases}$$

But, we will use the former form in this paper.

Secondly, since $J'>J$, it is only needed to prove that when q goes to q_0 the state $|JJ\rangle$ coincides with the state $|J'J\rangle$ up to a constant, i.e., when q goes to q_0 the following ratio B_m should be independent of m:

$$B_m \equiv \frac{(C_q^{j_1j_2})_{m(J-m)J'J}}{(C_q^{j_1j_2})_{m(J-m)JJ}}. \tag{23}$$

The coincidence of the rest states can be proved by the lowering operator f.

At last, we introduced a factor A_J in (19) in order to avoid divergence or all vanish of qCG coefficients. A_J is defined as follows:

$$A_J = q^{(J-j_1-j_2)(J-j_1+j_2+1)/2}\Delta(j_1j_2J)\left\{\frac{[2J+1]!}{[2j_1]![2j_2]!}\right\}^{1/2}\frac{[J'+j_2-j_1]![j_1+j_2-J']!}{[J+j_2-j_1]!} \tag{24}$$

$$(C_q^{j_1j_2})_{m(J-m)JJ} = (-1)^{j_1-m}q^{(j_1-m)(J+1)}\frac{[j_1+m]![j_2+J-m]!}{[J+j_1-j_2]![2j_2]!}\begin{bmatrix}2j_2\\ J'+j_2-j_1\end{bmatrix} \tag{25a}$$

and those obtained by replacing $J' \leftrightarrow J$, where the relation between J and J' is given in (22). We assume $J'=J$ when $\eta=(p-1)/2$ or $\eta=p-1/2$. Also, we have

$$(C_q^{j_1 j_2})_{m(J-m)J'J} = q^{-j_1(j_1+j_2+J-J')+m(j_1+j_2)} \frac{[J'+j_2-j_1]![j_1+m]![j_2+J-m]!}{[J'+J]![J+J_2-j_1]![j_1+j_2-J]!} \times \sum_n (-1)^n q^{n(J'+j_1+j_2+1)} \begin{bmatrix} J'+J \\ j_2+J-m-n \end{bmatrix} \begin{bmatrix} j_1-m \\ n \end{bmatrix} \times \begin{bmatrix} j_2-J+m \\ j_1+j_2-J'-n \end{bmatrix} \tag{25b}$$

$$n=\max\left\{\begin{matrix} 0 \\ j_1+J-J'-m \end{matrix}\right\} \ldots \min\left\{\begin{matrix} j_1-m \\ j_1+j_2-J' \end{matrix}\right\}.$$

Since $J-j_2 \leqslant m \leqslant j_1$, we have

$$J+j_2-j_1 \leqslant J+j_2-m-n \leqslant J'+j_2-j_1$$
$$j_2+J'-j_1=(t-s)p+\eta-\gamma$$
$$j_2+J-j_1=(l-t-s)p-\eta-\gamma-1.$$

In other words, condition (22) guarantees that there is a common r in the following three formulas

$$j_2+J'-j_1=rp+\beta_1 \qquad j_2+J-j_1=rp+\beta_0$$
$$j_2+J-m-n=rp+\beta \tag{26a}$$

$$0 \leqslant \beta_0 \leqslant \beta \leqslant \beta_1 < p \qquad r=\begin{cases} t-s & \text{if } \eta>\gamma \\ t-s-1 & \text{if } \eta<\gamma. \end{cases} \tag{26b}$$

It is the key point for proving the theorem that there exists a common r which is independent of m and n. Since $\beta_1-\beta_0=J'-J$, from (26) we have

$$0<J'-J \equiv \alpha < p. \tag{27}$$

Because $J+j_1-j_2=J'+J-(J'+j_2-j_1)=(l-r)p-\beta_1-1$, and when q goes to q_0

$$\begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix}_0 \neq 0 \qquad \text{if } \beta \leqslant \omega$$

$$\begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix} \sim [p] \qquad \text{if } \beta_1 > \omega$$

the non-vanishing components of $(C_q^{j_1 j_2})_{m(J-m)JJ}$ are finite and have the following values of m:

$$(n-1)p+\omega < m-J+j_2 \leqslant np+\beta_1 \qquad \text{if } \beta_1 \leqslant \omega$$
$$np+\omega < m-J+j_2 \leqslant np+\beta_1 \qquad \text{if } \beta_1 > \omega. \tag{28}$$

Except for the case $\omega=\beta_1$, those m satisfying (28) are separated into several groups with the higher bound m_h and the lower bound m_l:

$$j_1+m_h+1=0 \quad \text{mod } p \qquad j_2+(J-m_l)+1=0 \quad \text{mod } p$$

i.e.,

$$e|j_1 m_h\rangle=0 \qquad e|j_2(J-m_l)\rangle=0. \tag{29}$$

For the case $\omega=\beta_1$, all the components $(C_q^{j_1 j_2})_{m(J-m)JJ}$ with $J-j_2 \leqslant m \leqslant j_1$, when $q=q_0$, are nonvanishing.

4. Proof of the theorem

In this section we are going to prove that when q goes to q_0 the ratio B_m defined in (23) is independent of m. From (25) we have

$$B_m=\sum_n B_{mn}=(-1)^{j_1-m}q^{j_1(J'-J)-(j_1-m)(j_1+j_2+J+1)}\begin{bmatrix}2j_2\\J+j_2-j_1\end{bmatrix}\begin{bmatrix}2j_2\\J'+j_2-j_1\end{bmatrix}^{-1}$$
$$\times\begin{bmatrix}J'+J\\J'+j_2-j_1\end{bmatrix}^{-1}\sum_n(-1)^n q^{n(j_1+j_2+J'+1)}\begin{bmatrix}j_1-m\\n\end{bmatrix}$$
$$\times\begin{bmatrix}j_2-J+m\\j_1+j_2-J'-n\end{bmatrix}\begin{bmatrix}J'+J\\j_2+J-m-n\end{bmatrix}$$
$$=(-1)^{rp+\beta_0}q^{j_1\alpha+(rp+\beta_0)lp}\begin{bmatrix}up+\omega\\rp+\beta_0\end{bmatrix}\begin{bmatrix}up+\omega\\rp+\beta_1\end{bmatrix}^{-1}\begin{bmatrix}lp-1\\rp+\beta_1\end{bmatrix}^{-1}$$
$$\times q^{-(j_1-m)(j_1+j_2-J')}\sum_n q^{n(j_1+j_2-J)}\begin{bmatrix}j_1-m\\n\end{bmatrix}\begin{bmatrix}j_2-J+m\\j_1+j_2-J'-n\end{bmatrix}$$
$$\times\left\{(-1)^{rp+\beta}q^{-(rp+\beta)lp}\begin{bmatrix}lp-1\\rp+\beta\end{bmatrix}\right\}. \tag{30}$$

From (9a) we have

$$\lim_{q\to q_0}\left\{(-1)^{rp+\beta}q^{-(rp+\beta)lp}\begin{bmatrix}lp-1\\rp+\beta\end{bmatrix}\right\}=(-1)^{rp}\lambda^{r(p-1)}\binom{l-1}{r}. \tag{31}$$

By making use of an identity [9, 10]

$$\sum_n q^{\pm\{n(u+v)-ru\}}\begin{bmatrix}u\\n\end{bmatrix}\begin{bmatrix}v\\r-n\end{bmatrix}=\begin{bmatrix}u+v\\r\end{bmatrix} \tag{32a}$$

we obtain

$$q^{-(j_1-m)(j_1+j_2-J')}\sum_n q^{n(j_1+j_2-J)}\begin{bmatrix}j_1-m\\n\end{bmatrix}\begin{bmatrix}j_2-J+m\\j_1+j_2-J'-n\end{bmatrix}=\begin{bmatrix}j_1+j_2-J\\j_1+j_2-J'\end{bmatrix}. \tag{33}$$

Therefore, when q goes to q_0 the ratio B_m tends to a limit B independent of m:

$$\lim_{q\to q_0}B_m=B \tag{34a}$$

$$B\equiv(-1)^{\beta_0}\lambda^{r(p-1)+(rp+\beta_0)l}q_0^{j_1\alpha}\binom{l-1}{r}\begin{bmatrix}rp+\beta_1\\rp+\beta_0\end{bmatrix}_0\begin{bmatrix}lp-1\\rp+\beta_1\end{bmatrix}_0^{-1}$$
$$=(-1)^{\alpha}\lambda^{\alpha(l+r)}q_0^{j_1\alpha}\begin{bmatrix}\beta_1\\\beta_0\end{bmatrix}_0\neq 0. \tag{34b}$$

If there is no common r in (26), i.e., $J'+j_2-j_1=rp+\beta_1$, but $J+j_2-j_1=(r-1)p+\beta_0$, (31) as well as $\lim_{q\to q_0}B_m$ would obviously depend on m such that $|J'J\rangle$ and $|JJ\rangle$ would not coincide to each other.

There is a problem in the above proof. Equation (34) is deduced from (31), where we neglected the term proportional to $(q-q_0)$. It is allowed only when

$$\begin{bmatrix}j_1+j_2-J\\j_1+j_2-J'\end{bmatrix}_0\neq 0. \tag{35}$$

Owing to (9) and

$$j_1+j_2-J=2j_2-(j_2+J-j_1)=(u-r)p+(\omega-\beta_0)$$
$$j_1+j_2-J'=(u-r)p+(\omega-\beta_1)$$

(35) holds only when $\omega \geqslant \beta_1$ or $\omega < \beta_0$. Now, we discuss the case

$$\beta_0 \leqslant \omega < \beta_1 \tag{36}$$

where

$$\begin{bmatrix} j_1+j_2-J \\ j_1+j_2-J' \end{bmatrix} \sim [p]. \tag{37}$$

For this case we separate the values of m, $J-j_2 \leqslant m \leqslant j_1$, into two groups: m_1 and m_2:

$$\begin{aligned} m_1 &= J-j_2+v_1p+\mu_1 \qquad && \omega<\mu_1\leqslant\beta_1 \\ m_2 &= J-j_2+v_2p+\mu_2 \qquad && \mu_2\leqslant\omega \text{ or } \mu_2>\beta_1. \end{aligned} \tag{38}$$

From (28) we know that when $q=q_0$, $(C_q^{j_1j_2})_{m_1(J-m_1)JJ}\neq 0$ and $(C_q^{j_1j_2})_{m_2(J-m_2)JJ}\sim[p]$. By making use of (32$a$) and the following identities [9, 10] repeatedly,

$$\sum_n q^{\pm(n(u+v)-ru)}\begin{bmatrix} u+n-1 \\ n \end{bmatrix}\begin{bmatrix} v+r-n-1 \\ r-n \end{bmatrix}=\begin{bmatrix} u+v+r-1 \\ r \end{bmatrix} \tag{32b}$$

$$\sum_n (-1)^n q^{\pm(n(v-u-r+1)+ru)}\begin{bmatrix} u \\ n \end{bmatrix}\begin{bmatrix} v-n \\ r-n \end{bmatrix}=\begin{bmatrix} v-u \\ r \end{bmatrix} \tag{32c}$$

we obtain

$$\begin{aligned} \sum_n (-1)^n q^{n(j_1+j_2+J'+1)} & \begin{bmatrix} j_1-m \\ n \end{bmatrix}\begin{bmatrix} j_2-J+m \\ j_1+j_2+J'-n \end{bmatrix}\begin{bmatrix} J'+J \\ j_2+J-m-n \end{bmatrix} \\ &= (-1)^{j_2+J-m} q^{(j_2+J-m)(J'+j_1+j_2+1)-(J'+J)(J'-j_1+j_2)} \\ &\quad \times\sum_n (-1)^n q^{-n(j_1+j_2-J'+1)}\begin{bmatrix} J'+J \\ n \end{bmatrix}\begin{bmatrix} j_1-m+n \\ j_2+J-m \end{bmatrix}\begin{bmatrix} J'+j_2+m-n \\ j_1+m \end{bmatrix} \\ &\quad rp+\beta_0\leqslant n\leqslant rp+\beta_1. \end{aligned} \tag{39}$$

For the case (36) and $m=m_1$ the last two factors in the right-hand side of (39) must contain a $[p]$ factor. Now, we are able to prove that for the case (36)

$$\begin{aligned} \lim_{q\to q_0} & (-1)^{j_1-m_1} q^{-(j_1-m_1)(j_1+j_2+J+1)} \frac{1}{[p]}\sum_n (-1)^n q^{n(j_1+j_2+J'+1)}\begin{bmatrix} j_1-m_1 \\ n \end{bmatrix} \\ &\times\begin{bmatrix} j_2-J+m_1 \\ j_1+j_2-J'-n \end{bmatrix}\begin{bmatrix} J'+J \\ j_2+J-m_1-n \end{bmatrix} \\ &= (-1)^{\alpha-\omega-1}\lambda^{r(p-1)+(rp+\beta_0)l+\alpha(u-r)+1}\begin{pmatrix} l-1 \\ r \end{pmatrix} \\ &\quad\times(u+l-r)[\omega-\beta_0]![\beta_1-\omega-1]!/[\alpha]! \\ &= (-1)^{\beta_0}\lambda^{r(p-1)+(rp+\beta_0)l}\begin{pmatrix} l-1 \\ r \end{pmatrix}\left(1+\frac{l}{u-r}\right) \\ &\quad\times\lim_{q\to q_0}\frac{1}{[p]}\begin{bmatrix} j_1+j_2-J \\ j_1+j_2-J' \end{bmatrix}. \end{aligned} \tag{40}$$

Therefore, the ratio B_m is also independent of m for the case (36):

$$\lim_{q\to q_0} B_{m_1} = \tilde{B} \qquad \tilde{B} = \left(1 + \frac{l}{u-r}\right) B$$
$$\lim_{q\to q_0} (C_q^{j_1j_2})_{m_2(J-m_2)JJ} = \lim_{q\to q_0} (C_q^{j_1j_2})_{m_2(J-m_2)J'J} = 0 \tag{41}$$

5. New states

In the previous sections we showed that under the condition (22) when q goes to q_0 the state $|JJ\rangle$ coincides with the state $|J'J\rangle$ up to a constant. Owing to this coincidence, some new states must exist in the linear space spanned by $|j_1m_1\rangle|j_2m_2\rangle$. The new state with the highest weight can be computed by a limit process

$$|JJ\rangle_0 = \sum_m |j_1m\rangle|j_2(J-m)\rangle(\bar{C}_{q_0}^{j_1j_2})_{m(J-m)JJ}$$

$$\begin{aligned}
N(\bar{C}_{q_0}^{j_1j_2})_{m(J-m)JJ}
&= \lim_{q\to q_0} \frac{(C_q^{j_1j_2})_{m(J-m)J'J} - B(C_q^{j_1j_2})_{m(J-m)JJ}}{q-q_0} \\
&= \lim_{q\to q_0} \frac{(B_m - B)(C_q^{j_1j_2})_{m(J-m)JJ}}{q-q_0} \\
&= \lim_{q\to q_0} \{(C_q^{j_1j_2})_{m(J-m)J'J}\}' - B\{(C_q^{j_1j_2})_{m(J-m)JJ}\}'
\end{aligned} \tag{42}$$

where B should be replaced by $\tilde{B}$ for the case (36). The definition (42) guarantees that the new state $|JJ\rangle_0$ is orthogonal to the states belonging to the other IRs

$$\langle J''J|JJ\rangle_0 = 0 \qquad J'' \neq J \text{ and } J' \tag{43}$$

and the normalization factor N can be determined by (44)

$$A_{J'}^2\langle J'J|JJ\rangle_0 = \frac{A_{J'}^2\langle J'(J+1)|J'(J+1)\rangle}{[J'-J]} = \frac{\lambda^{l(\alpha-1)}}{[\alpha]} \tag{44}$$

such that

$$e|JJ\rangle_0 = |J'(J+1)\rangle. \tag{45a}$$

The rest new states $|JM\rangle_0$ can be computed by [6]

$$|JM\rangle_0 = \frac{f}{[J-M]}|J(M+1)\rangle_0 \qquad -J \leq M < J. \tag{45b}$$

It is easy to show from the quantum algebraic relations of $U_q\,sl(2)$ that

$$\begin{aligned}
&h|JM\rangle_0 = 2M|JM\rangle_0 \\
&e|JM\rangle_0 = [J+M+1]|J(M+1)\rangle_0 + \begin{bmatrix} J'-M-1 \\ J-M \end{bmatrix}|J'(M+1)\rangle \\
&f|J-J\rangle_0 = \begin{bmatrix} J'+J \\ J'-J \end{bmatrix}|J'-(J+1)\rangle.
\end{aligned} \tag{45c}$$

At last, we are going to discuss the method of computing $(\bar{C}_{q_0}^{j_1j_2})_{m(J-m)JJ}$ for the three different cases.

(i) $\omega \geqslant \beta_1$

In this case both

$$\begin{bmatrix} 2j_2 \\ J+j_2-j_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix}$$

are non-vanishing when $q=q_0$, but some B_{mn} may be vanishing. When it occurs, the summation over n is separated into two parts: n_1 and n_2, where $B_{mn_1} \neq 0$ and $B_{mn_2} \sim [p]$ when q goes to q_0.

If m does not satisfy (28), $(C_q^{j_1j_2})_{m(J-m)JJ}$ is proportional to $[p]$ so that

$$N(\bar{C}_{q_0}^{j_1j_2})_{m(J-m)JJ} = \lim_{q\to q_0} \frac{(B_m - B)(C_q^{j_1j_2})_{m(J-m)JJ}}{q-q_0} = 0. \tag{46}$$

However, for those m satisfying (28), from (42) we have

$$\begin{aligned} N(\bar{C}_{q_0}^{j_1j_2})_{m(J-m)JJ} &= \lim_{q\to q_0} \{(C_q^{j_1j_2})_{m(J-m)JJ}\} \frac{\mathrm{d}}{\mathrm{d}q} B_m \\ &= \lim_{q\to q_0} \{(C_q^{j_1j_2})_{m(J-m)J'J}\} B^{-1} \frac{\mathrm{d}}{\mathrm{d}q} \sum_n B_{mn} \end{aligned} \tag{47}$$

and $B^{-1}\,\mathrm{d}/\mathrm{d}q\, B_m$ can be calculated by (15):

$$\begin{aligned} \lim_{q\to q_0} B^{-1} \frac{\mathrm{d}}{\mathrm{d}q} B_m &= \{j_1(J'-J)-(j_1-m)(j_1+j_2+J+1)\}q_0^{-1} + \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} 2j_2 \\ J+j_2-j_1 \end{bmatrix} \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix} - \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} J'+J \\ J'+j_2-j_1 \end{bmatrix} \\ &\quad + B^{-1} \sum_{n_1} B_{mn_1} \left\{ n_1(j_1+j_2+J'+1)q_0^{-1} + \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} J'+J \\ J+j_2-m-n_1 \end{bmatrix} \right. \\ &\quad \left. + \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} j_1-m \\ j_1-m-n_1 \end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}q} \ln \begin{bmatrix} j_2-J+m \\ J'-J-j_1+m+n_1 \end{bmatrix} \right\} \\ &\quad + B^{-1} \frac{2p\lambda}{q_0^2-1} \sum_{n_2} \frac{B_{mn_2}}{[p]}. \end{aligned} \tag{48}$$

After the derivative q goes to q_0.

(ii) $0 \leqslant \omega < \beta_0$

In this case both

$$\begin{bmatrix} 2j_2 \\ J+j_2-j_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix}$$

are vanishing when $q=q_0$. The formulas (46), (47) and (48) will hold for this case except for the second and the third terms on the right-hand side of (48) which should be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}q} \ln \left\{ \begin{bmatrix} 2j_2 \\ J+j_2-j_1 \end{bmatrix} \Big/ [p] \right\} - \frac{\mathrm{d}}{\mathrm{d}q} \ln \left\{ \begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix} \Big/ [p] \right\}$$

(iii) $\beta_0 \leqslant \omega < \beta_1$

In this case

$$\begin{bmatrix} 2j_2 \\ J+j_2-j_1 \end{bmatrix}$$

is non-vanishing but

$$\begin{bmatrix} 2j_2 \\ J'+j_2-j_1 \end{bmatrix}$$

is vanishing when $q = q_0$. Because $\lim_{q\to q_0}(B_{m_2} - \tilde{B})$ is, generally, no longer vanishing, we have

$$\begin{aligned} N(\tilde{C}_{q_0}^{j_1j_2})_{m_2(J-m_2)JJ} &= \frac{2p\lambda}{q_0^2 - 1} \lim_{q\to q_0} (B_{m_2} - \tilde{B})(C_q^{j_1j_2})_{m_2(J-m_2)JJ}/[p] \\ N(\tilde{C}_{q_0}^{j_1j_2})_{m_1(J-m_1)JJ} &= \lim_{q\to q_0} (C_q^{j_1j_2})_{m_1(J-m_1)JJ} \frac{\mathrm{d}}{\mathrm{d}q} B_{m_1} \end{aligned} \tag{49}$$

where (39) is helpful for calculating $\mathrm{d}/\mathrm{d}q\, B_{m_1}$.

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